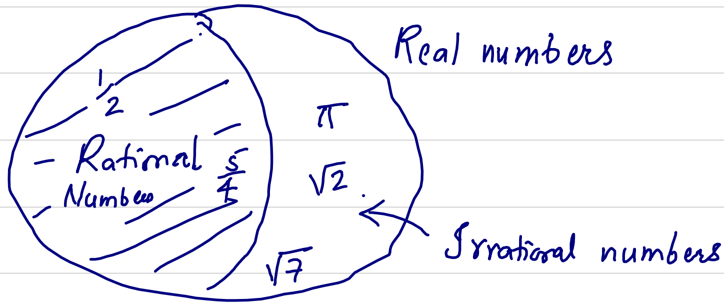
Rational Fuctions

Recall some facts about rational numbers:

$$\text{Rational number} = \frac{\text{Integer}}{\text{Integer}}$$

Ex: $\frac{2}{3}$, $\frac{-1}{5}$, 0.00175, $0.111\ldots$

Note that not all real numbers are rationals. Rationals are a proper subset of real numbers. Ex. $\sqrt{2}$ is not rational.



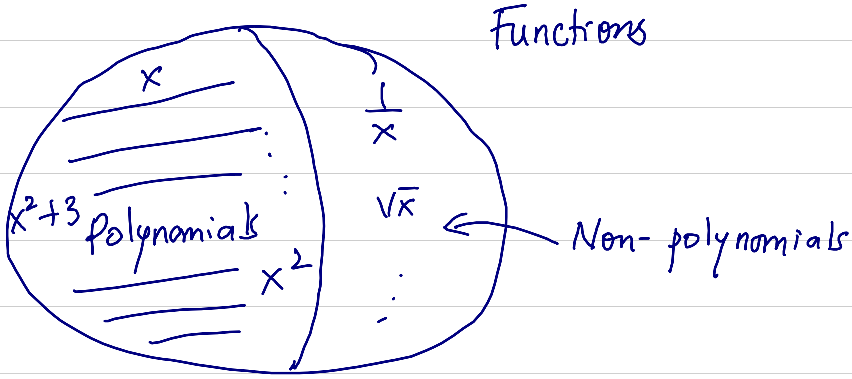
We follow the above analogy to define rational functions:

Recall that $\text{Polynomial} = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$

$a_i \in \mathbb{R}$ for $0 \le i \le n$.

Note that not all functions are polynomials. For ex.,

$\frac{1}{x}$, $\sqrt{x}$, $\frac{x^2+3}{x}$ are not polynomials.



Def. A function $f(x)$ is rational if
$$f(x) = \frac{n(x)}{d(x)}$$
where the numerator $n(x)$ and the denominator $d(x)$ are polynomial functions.

Note that for $f(x)$ to be well defined $d(x)$ must be nonzero. Thus, Domain of $f = \{$ All x in domain of n and domain of d but with $d(x) \neq 0 \}$

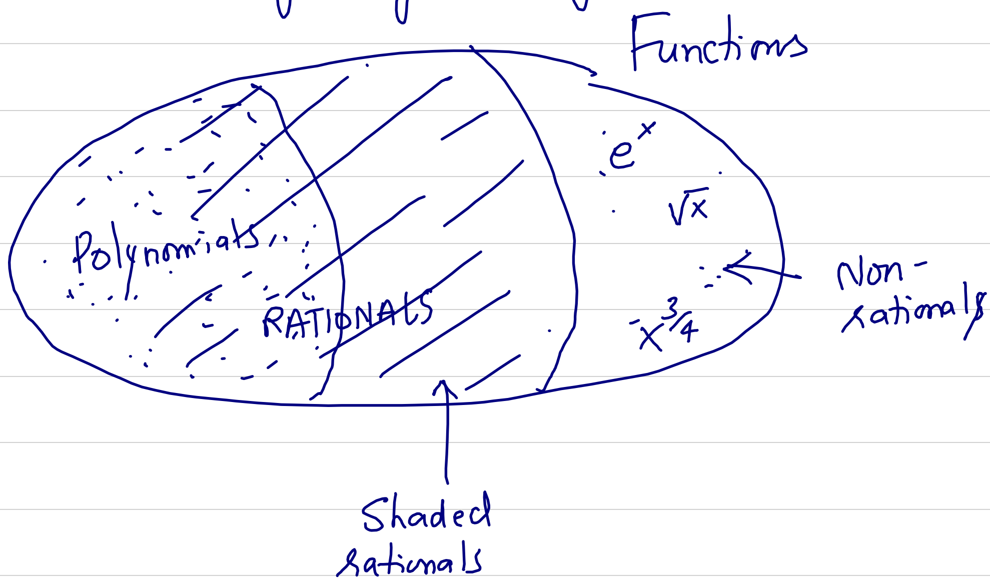
Examples:
$$\frac{2x^2 - 3x + 1}{x^5 + 7}, \quad \frac{4x^3 + x + 1/2}{7.5x + 2}, \quad \frac{44x^{100}}{7.9x^6 - 3},$$
$$\frac{68x^{1001} - x^{10} + 7}{37}$$

Consider a special case: $d(x)$ is a constant number. In this case we have
$$f(x) = \frac{n(x)}{\text{number}}$$

Ex. $$\frac{75x^7 - 3x + 4}{6}$$
$$= \frac{75}{6}x^7 - \frac{3}{6}x + \frac{4}{6}$$ (We can break it up)

But this is a polynomial. So when $d(x) =$ constant, $f(x)$ is a polynomial. Hence, all polynomials are rational functions.

Thus, we have the following set diagram:



Note Polyomials are contained inside rational functions.

In this chapter, we are interested in analyzing rational functions. We want to ask questions:

1) What's the domain of ratimal functions?
2) What's the range " " "
3) " " X-intercept / zeros of ratimal functions?
4) " " Y-intercept " " "
5) What does the graph look like?

Problems:

1) Find the domain of $f(x) = \dfrac{x+1}{x^2-x-6}$. Express the domain in interval notation.

Solution. We want to know when the denominator is zero and avoid those numbers.

$$x^2 - x - 6 = 0$$

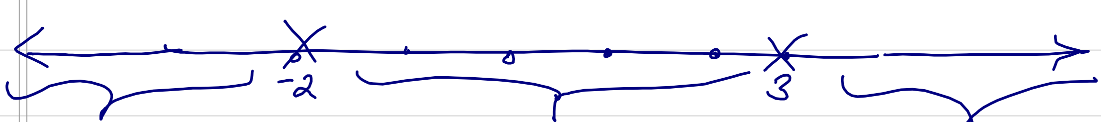
$$(x+2)(x-3) = 0$$

$$x = -2 \quad \text{or} \quad x = 3.$$

$2 - 3 = -1$

$2 \cdot (-3) = -6$

Thus, we need to avoid $x = -2$ and $x = 3$.



Domain: $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$

(2) Exercise:

Find the domain of

$$f(x) = \frac{x-2}{x^2-3x-4}$$

(3) Find the domain of $g(x) = \dfrac{3x}{x^2+9}$.

Solution. When is the denominator zero?

When $x^2+9=0$

$$\Rightarrow x^2 = -9$$

$$\Rightarrow x = \pm\sqrt{-9}$$

$$\Rightarrow x = \pm\sqrt{9}\,\sqrt{-1}$$

$$\Rightarrow x = \pm 3i$$

There are only imaginary solutions. So the denominator is nonzero for any real number x.

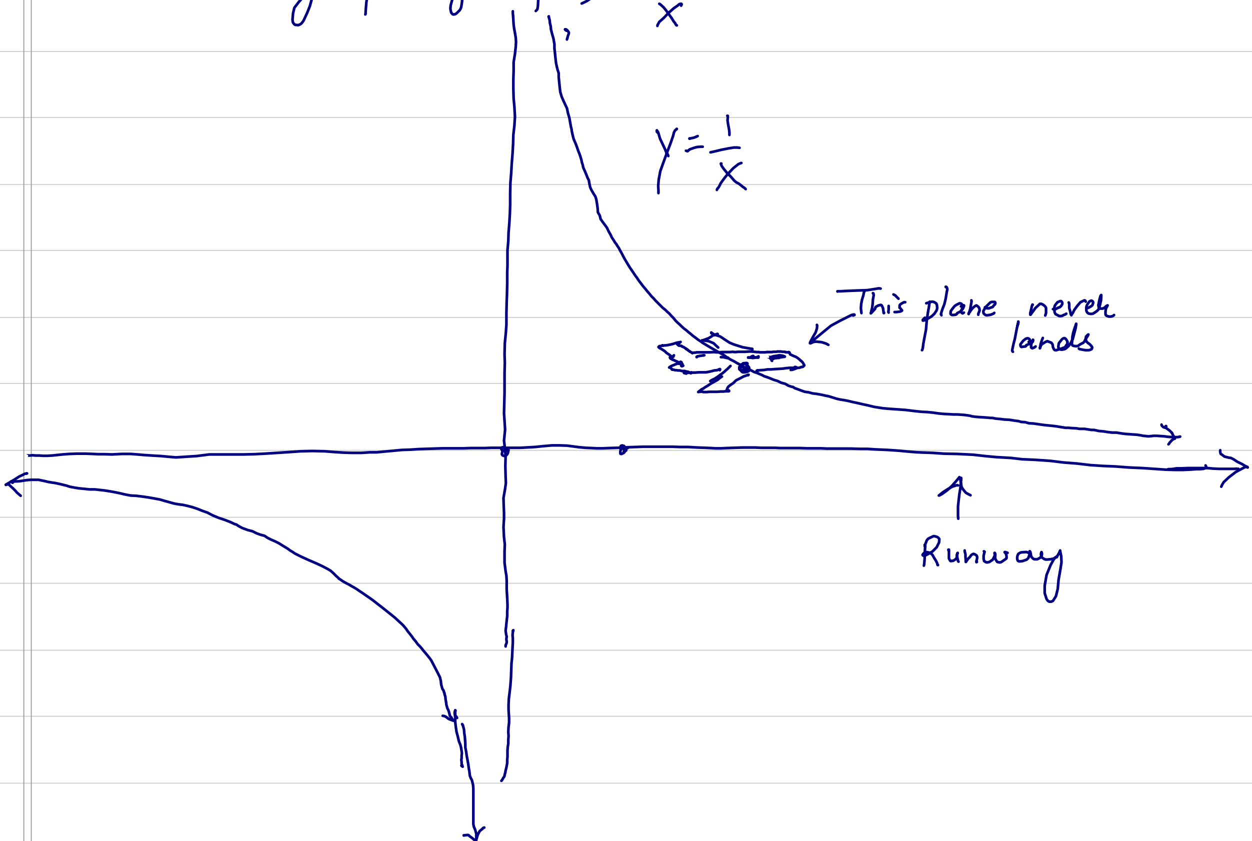
Thus domain is $(-\infty,\infty)$. $\square$

Alternatively, you could argue that since x^2 is always nonnegative (≥ 0), x^2+9 is always positive (>0).

Asymptotes

Graphs of rational functions have asymptotes which polynomials do not have.

Recall the graph of $f(x) = \frac{1}{x}$



The line $y=0$ is a horizontal asymptote.
The line $x=0$ is a vertical asymptote.

How?

Let x be a very large number. Say $x = 138$ billion (Jeff Bezos net worth).

Then $f(x) = \frac{1}{x} = \frac{1}{138 \text{ billion}}$ which is very small.

Now let x be a very small number. Say $x = \frac{1}{138 \text{ billion}}$. Then $f(x) = \frac{1}{x} = \frac{1}{\frac{1}{138b.}} = 138\ b.$ which is very large.

SOME NONSENSICAL EQUATIONS THAT NEED TO BE INTERPRETED APPRORIATELY.

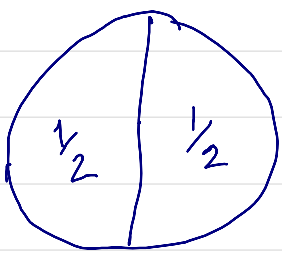
Ques. How do you interpret $\frac{8}{4} = 2$?

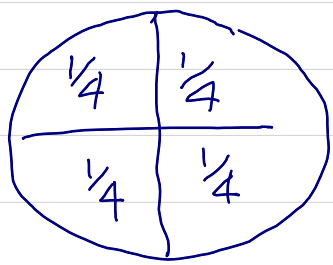
Ans. If we divide 8 candies among 4 people, each one gets 2 candies.

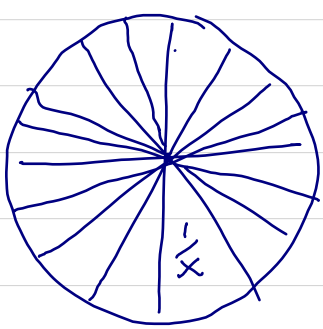
Let's try to make sense of the following:

$$\frac{1}{\infty} = 0 \quad , \quad \frac{1}{-\infty} = 0 \quad , \quad \frac{1}{0} = \pm\infty$$

Pizza







When x is very large.

so as $x \to \infty$, $\frac{1}{x} \to 0$.

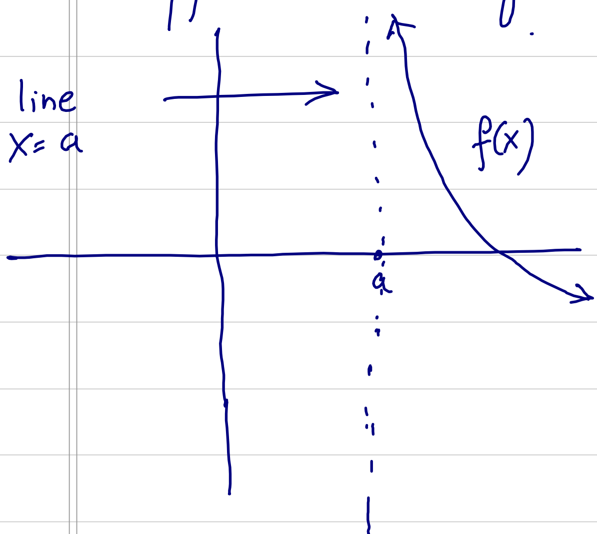
when x is very small. Say $x = \frac{1}{10^{10}}$. Then

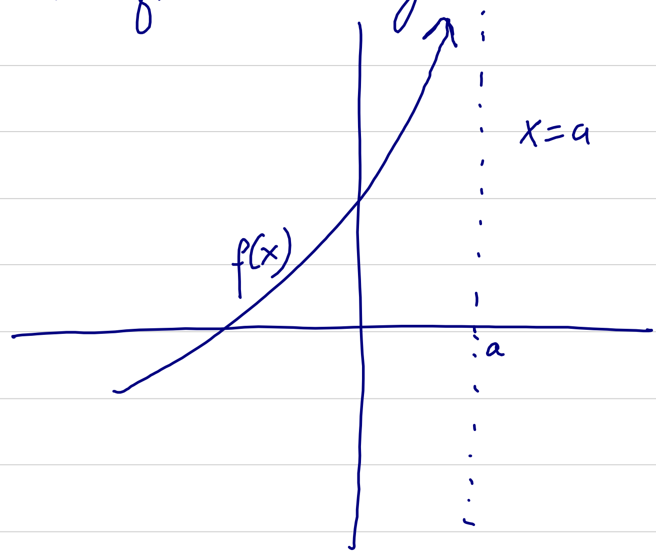
$$\frac{1}{x} = \frac{1}{\frac{1}{10^{10}}} = 10^{10}$$ which is very large.

Formal definition

Vertical asymptote

Def. The line $x = a$ is a vertical asymptote for the graph of a function if $f(x)$ either increases or decreases without bound as x approaches a from either left or the right.



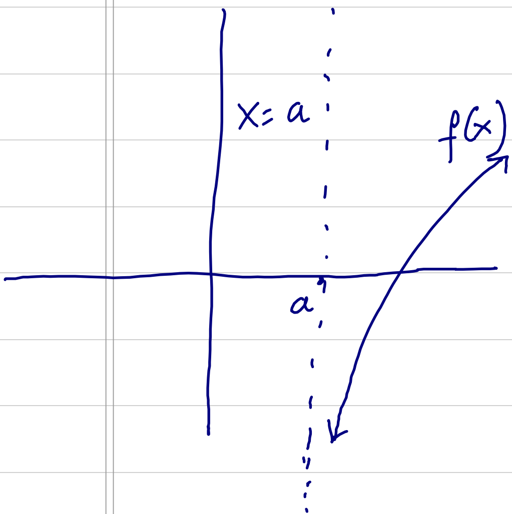


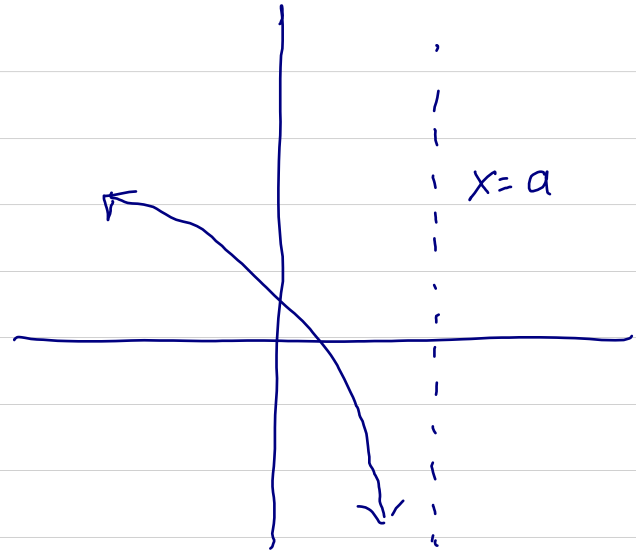
As $x \to a^+$, $f(x) \to \infty$

this means you are approaching a from the right

As $x \to a^-$, $f(x) \to \infty$.

This means you are approaching from the left





As $x \to a^+$,
$f(x) \to -\infty$

As $x \to a^-$,
$f(x) \to -\infty$.

<u>Question</u>. How to locate the vertical asymptotes of a rational function?

<u>Ans</u>. Let $f(x) = \dfrac{n(x)}{d(x)}$

<u>Step 1</u>. factor the numerator and denominator whenever possible.

<u>Step 2</u>. Cancel out the common factor.
But remember that the canceled zero cannot be in the domain (the den. cannot be zero).

<u>Step 3</u>. Find the values for which the denominator is zero.

These zeros are your vertical asymptotes.

Examples:

Locate the vertical asymptotes of

$$f(x) = \frac{5x+2}{6x^2 - x - 2}$$

Solution. Step 1 (Factor denominator)

$$6x^2 - x - 2$$
$$= 6x^2 - 4x + 3x - 2$$
$$= 2x(3x-2) + 1\cdot(3x-2)$$
$$= (3x-2)(2x+1)$$

$6 \cdot -2 = -12$

$(-4) + 3 = -1$

$(-4) \cdot 3 = -12$

Thus, $f(x) = \dfrac{5x+2}{(3x-2)(2x+1)}$

Step 2 Cancel any common factors.
(None)

Step 3. When is the denominator zero?

$3x - 2 = 0$ and $2x + 1 = 0$

$x = 2/3$ and $x = \dfrac{-1}{2}$.

The vertical asymptotes are $x = \dfrac{2}{3}$ and $x = \dfrac{-1}{2}$.

Exercises

1) Locate vertical asymptotes for

(a) $f(x) = \dfrac{3x-1}{2x^2 - x - 15}$

(b) $h(x) = \dfrac{1 - x^2 + x^3}{2x^6 + 3x^4}$

(c) $f(x) = \dfrac{2x}{2x^4 + 15x^2 + 25}$

Ex. Locate any vertical asymptote for

$$f(x) = \frac{x+2}{x^3-3x^2-10x}$$

Solution. Step 1. Factor $d(x)$:

$$x^3-3x^2-10x$$
$$= x(x^2-3x-10)$$
$$= x(x-5)(x+2)$$

Thus, $$f(x) = \frac{x+2}{x(x-5)(x+2)}$$

Step 2: Cancel common factors

$$f(x) = \frac{\cancel{x+2}}{x(x-5)\cancel{(x+2)}}$$

$$= \frac{1}{x(x-5)}$$

NOTE: ALTHOUGH WE CANCELLED $(X+2)$, YOU MUST KNOW THAT THE DOMAIN WILL NOT INCLUDE $X=-2$, SINCE SETTING $X=-2$ WILL IMPLY THAT THE DENOMINATOR IS 0.

Step 3. When is denominator zero after cancelling?

$\boxed{x=0}$ and $x-5=0$

$\Rightarrow \boxed{x=5}$

The vertical asymptotes are $x=0$ and $x=5$.

IN THIS CASE $X=-2$ IS CALLED A HOLE. THE FUNCTION IS NOT DEFINED FOR $X=-2$.

Exercise:

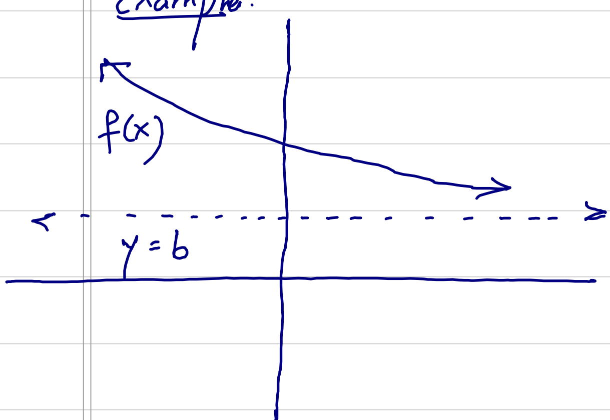
Locate the domain & vertical asymptotes of

$$f(x) = \frac{x^2 - 4x}{x^2 - 7x + 12}$$

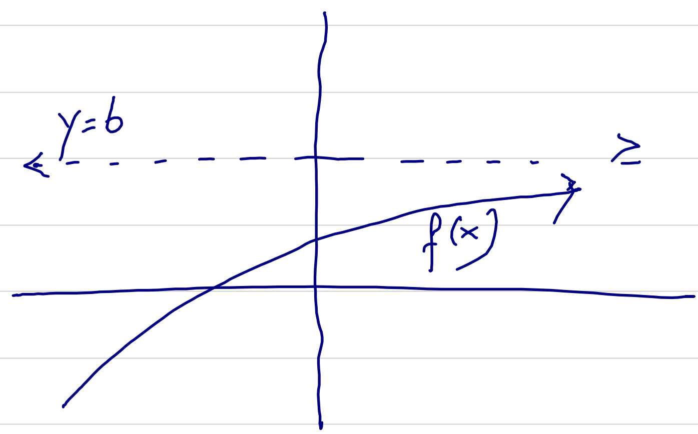
Horizontal Asymptote

Def. The line $y = b$ is a horizontal asymptote of the graph of a function if $f(x)$ approaches b as x increases or decreases without bound.

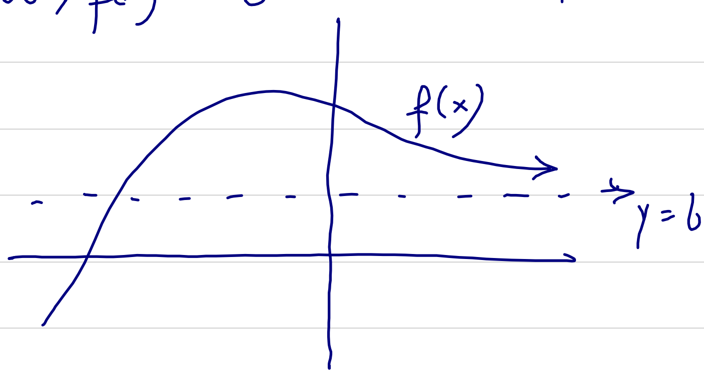
Example:



As $x \to \infty$, $f(x) \to b$



As $x \to \infty$, $f(x) \to b$.



As $x \to \infty$, $f(x) \to b$

Ques. How to locate the horizontal asymptote?

Ans. Let f be a rational function.

Then $f(x) = \frac{n(x)}{d(x)}$ where $n(x)$ and $d(x)$ are polynomials.

Let $n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$.

Let $d(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0$.

Then $f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0}$.

1. When $n < m$ the X-axis $(y=0)$ is horizontal asymptote.
2. When $n = m$, the line $y = \frac{a_n}{b_m}$ is the horizontal asymptote.
3. When $n > m$ there is no horizontal asymptote.

Ques. Why?

Let's prove 1:

Say that $n < m$. We know that as x gets very large, i.e., as $x \to \infty$, $n(x)$ behaves like the power function $a_n x^n$, while $d(x)$ behaves like the power function $b_m x^m$. So when $x \to \infty$, $f(x)$ behaves like $\frac{a_n x^n}{b_m x^m}$.

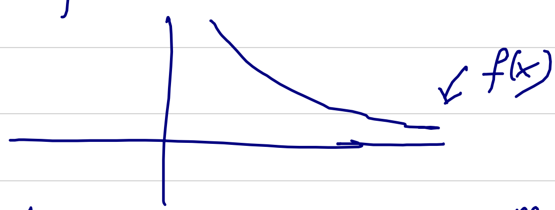
$$= \frac{a_n}{b_m} \cdot \frac{1}{x^{m-n}}$$

Note $m > n$.

So $f(x)$ behaves like $\frac{a_n}{b_m} \cdot \frac{1}{x^{m-n}}$

and $m-n$ is positive.
So when x is very large x^{m-n} is even larger.
But $\frac{1}{x^{m-n}}$ is very small. $\frac{a_n}{b_m}$ is just a

constant. So when $x \to \infty$, $f(x)$ is very small, i.e. $f(x) \to 0$.



Similarly when $x \to -\infty$, $x^{m-n} \to -\infty$ or $x^{m-n} \to \infty$ depending on whether $m-n$ is even or odd. But $\frac{1}{\infty} = \frac{1}{-\infty} = 0$.

So when $x \to -\infty$, $f(x)$ is very small.

(2) Case: $n = m$.
When $x \to \infty$, $f(x)$ behaves like $\frac{a_n x^n}{b_m x^n} = \frac{a_n}{b_m}$.

Thus, as $x \to \infty$, $f(x) \to \infty$. Similarly as $x \to -\infty$, $f(x)$ behaves like $\frac{a_n}{b_m}$

(3) Case $n > m$.
When $x \to \infty$, $f(x)$ behaves like $\frac{a_n x^n}{b_m x^m} = \frac{a_n}{b_m} x^{n-m}$

Note $n-m > 0$ as $n > m$.

Thus as $x \to \infty$ $f(x)$ behaves like $\frac{a_n}{b_m} x^{n-m}$.

When $x \to \infty$, $x^{n-m} \to \infty$. So there is no horizontal asymptote. Similar argument works for $x \to -\infty$.

Examples:

(a) $f(x) = \dfrac{8x+3}{4x^2+1}$

The degree of numerator $8x+3$ is 1 $\quad n = 1$

The degree of denominator $4x^2+1$ is 2 $\quad m = 2$.

The degree of num. is less than the degree of denom. $\quad n < m$

Thus, the X-axis is the horizontal asymptote for $f(x)$

$\underset{y=0}{\parallel}$

(b) $g(x) = \dfrac{8x^2+3}{4x^2+1}$

The degree of numerator is 2.

The degree of denominator is 2

The ratio of leading coefficients $\dfrac{a_n}{b_m} = \dfrac{8}{4} = 2$.

Thus, the line $y = 2$ is the horizontal asymptote.

(c) $h(x) = \dfrac{8x^3+3}{4x^2+1}$

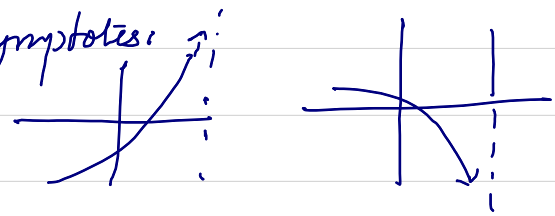
The degree of num. is 3 $\quad n = 3$
The degree of denom. is 2 $\quad m = 2$
Since $n > m$, there is no horizontal asymptote.
But in this case there is a slant asymptote.

Exercise. Find the horizontal asymptote (if one exists) for the graph of the rational function

$$f(x) = \frac{7x^3 - x^2 - 2}{-4x^3+1}.$$

There are 3 types of asymptotes:

1) Vertical asymptote



2) Horizontal asymptote

3) Slant asymptote

<u>Slant Asymptotes</u>

Let f be a rational function given by $f(x) = \frac{n(x)}{d(x)}$,

where $n(x)$ and $d(x)$ are polynomials. and the degree of $n(x)$ is one more than the degree of $d(x)$.

Then $f(x) = \underbrace{mx+b}_{\text{quotient}} + \frac{r(x)}{d(x)}$

where the degree of the remainder $r(x)$ is less than the degree of $d(x)$.

The line $y = mx+b$ is the <u>slant asymptote</u>.

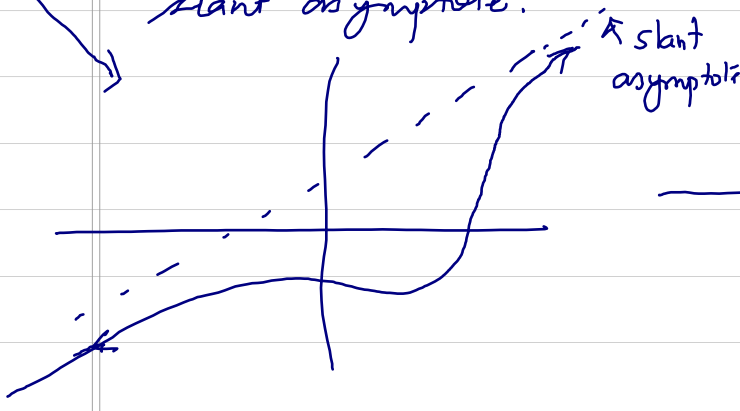
As $x \to -\infty$ or $x \to \infty$, $f(x) \to mx+b$.

<u>Example:</u>

Find the slant asymptote of the rational function

$$f(x) = \frac{4x^3+x^2+3}{x^2-x+1}$$

<u>Soln</u>. Note the degree of numerator is 1 more than the degree of denominator. So there is a slant asymptote.



$$\begin{array}{r} 4x+5 \\ x^2-x+1 \,\big)\, \overline{4x^3+x^2+3} \\ \underline{\overset{(-)}{4x^3} \overset{(+)}{-4x^2} \overset{(-)}{+4x}} \\ 5x^2-4x+3 \\ \underline{\overset{(-)}{5x^2} \overset{(+)}{-5x} \overset{(-)}{+5}} \\ x-2 \end{array}$$

Thus, $f(x) = 4x+5 + \dfrac{x-2}{x^2-x+1}$

Note that as $x \to \pm\infty$, $\dfrac{x-2}{x^2-x+1} \longrightarrow 0$.

The line $\boxed{y = 4x+5}$ is the slant asymptote.

Exercise

Find the slant asymptote of the rational functions.

(a) $f(x) = \dfrac{x^2+3x+2}{x-2}$

(b) $f(x) = \dfrac{3x^3+2x^2+3x+4}{x^2+1}$

(b) $f(x) = \dfrac{3x^3 + 2x^2 + 3x + 4}{x^2+1}$

Soln. Note that degree of numerator is one more than the degree of denominator.

we have

$$
\begin{array}{r|l}
x^2+1 & 3x^3 + 2x^2 + 3x + 4 \quad (3x+2 \\
 & 3x^3 + \quad\ \ + 3x \\
 & (-) \qquad\quad\ \ (-) \\
\hline
 & \qquad 2x^2 + 4 \\
 & \qquad 2x^2 + 2 \\
 & \qquad (-) \quad (-) \\
\hline
 & \qquad\qquad 2
\end{array}
$$

Thus, $f(x) = 3x+2 + \dfrac{2}{x^2+1}$

Note that as $x \to \pm\infty$, $\dfrac{2}{x^2+1} \to 0$. Thus,

the line $y = 3x+2$ is the slant asymptote.

Graphing Rational Functions

Let f be a rational function given by $f(x) = \frac{n(x)}{d(x)}$

Step 1. Find the domain of f.

Step 2. Find the X-intercepts & Y-intercepts

Step 3. Find any holes:

- Factor the numerator and denominator.
- Divide out common factors.
- A common factor $x-a$ corresponds to a hole on the graph of f at $x=a$ if the multiplicity of a in the numerator is greater than or equal to the multiplicity of a in the denominator

The result after cancelling common factors is

$$R(x) = \frac{p(x)}{q(x)} \text{ in lowest terms}$$

Step 4 Find any asymptotes

Vertical : solve $q(x) = 0$

Horizontal: Compare the degrees

$\left\{ \text{Slant : Provided } p(x) \text{ has 1 degree more than } q(x). \right.$

Step 5: Find additional points near asymptotes

Step 5: Sketch.

Ex. Graph the rational function $f(x) = \frac{x}{x^2-4} = \left(\frac{n(x)}{d(x)}\right)$

Soln. Step 1. (Domain)

when is $x^2-4=0$?

$$x^2-4=0$$
$$\Rightarrow x^2=4$$
$$\Rightarrow x=\pm 2.$$

Thus, the domain is $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$.

Step 2 (Intercepts)

Y-intercept: $f(0) = \frac{0}{0^2-4}$

$= -\frac{1}{4}$ $\Rightarrow$ $(0,0)$ is Y-int.

X-intercept: Solve $f(x) = 0$

$$\frac{x}{x^2-4} = 0$$

$\Rightarrow$ $x=0.$ $\Rightarrow$ $(0,0)$ is X-int.

Step 3 (Holes)

$$d(x) = x^2-4$$
$$= (x+2)(x-2)$$

We cannot cancel any common factors at this time.

Thus, $f(x) = \frac{x}{(x+2)(x-2)}$.

There are no holes.

Step 4 (Asymptotes)

Vertical Asymptote: $d(x) = (x+2)(x-2) = 0$

$\boxed{x = -2}$ $\boxed{x = 2}$

Horizontal Asymptotes:

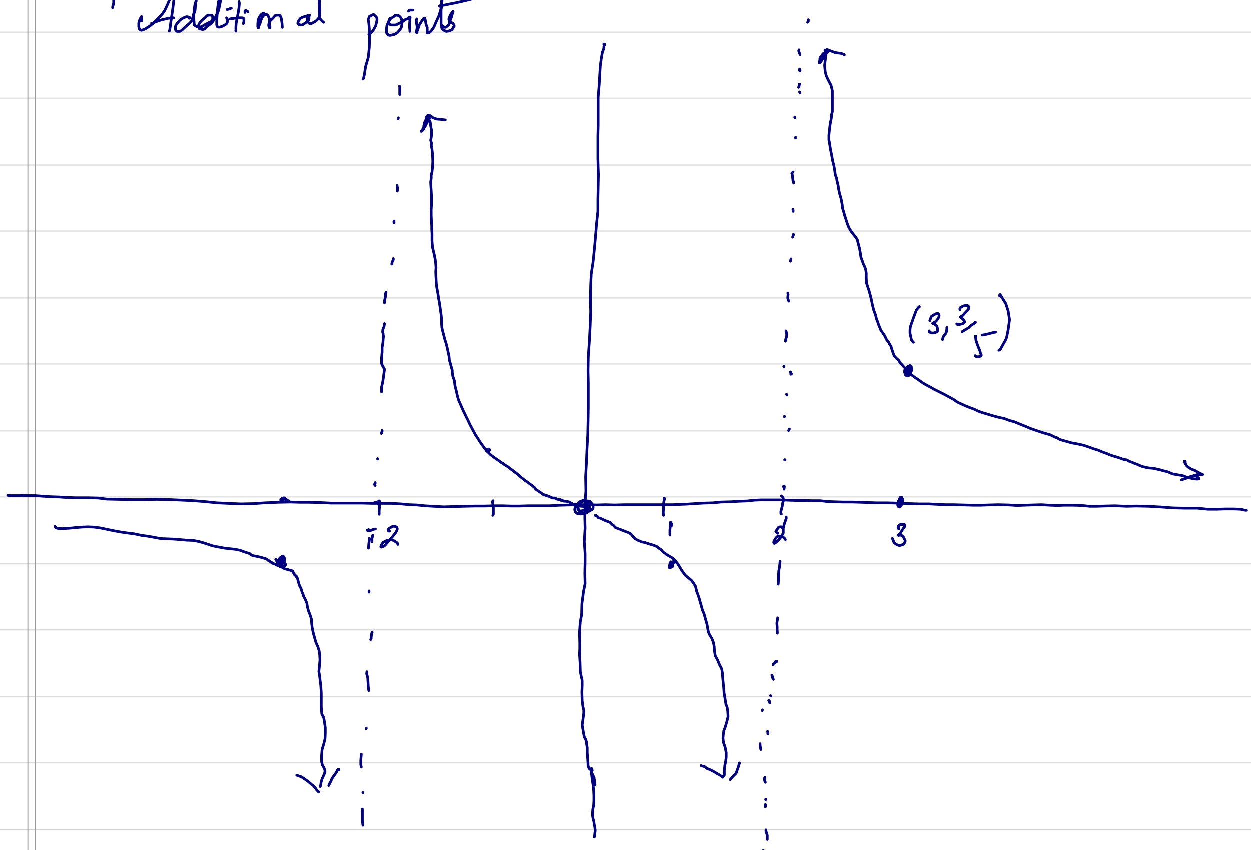
$$\frac{\text{Degree of numerator}}{\text{Degree of denominator}} = \frac{1}{2}$$

Since degree of numerator (n) < degree of denominator (m), $y = 0$ (X-axis) is a horizontal asymptote

Slant Asymptote Not applicable

Step 5

Additional points



x	3	1	-1	-3
y	$3/5$	$-1/3$	$1/3$	$-3/5$

Exercise:

Graph the function $f(x) = \dfrac{x}{x^2-1}$

Ex. State the asymptotes (if any) and graph the rational function $f(x) = \dfrac{x^4 - x^3 - 6x^2}{x^2-1}$

Soln. Step 1 (Domain)

$$x^2 - 1 = 0$$
$$\Rightarrow x^2 = 1$$
$$\Rightarrow x = \pm 1.$$

Thus, domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

Step 2 (Intercepts)

Y-intercept: $f(0) = \dfrac{0}{-1} = 0 \Rightarrow (0,0)$ is Y-int.

X-intercept: $f(x) = 0$

$$\frac{x^4 - x^3 - 6x^2}{x^2-1} = 0$$

$$\Rightarrow x^4 - x^3 - 6x^2 = 0$$
$$\Rightarrow x^2(x^2 - x - 6) = 0$$
$$\Rightarrow x^2(x-3)(x+2) = 0$$
$$\Rightarrow x = 0, \quad x = 3 \text{ and } x = -2.$$

Thus, the X-intercepts are $(0,0)$, $(3,0)$ and $(-2,0)$.

Step 3 (Holes).

we have $n(x) = x^2(x-3)(x+2)$,

$d(x) = x^2-1$

$= (x-1)(x+1)$.

Thus, $f(x) = \dfrac{x^2(x-3)(x+2)}{(x-1)(x+1)}$

- There are no common factors so there are no holes.

Step 4 (Asymptotes)

Vertical $\rightarrow$ $(x-1)(x+1) = 0$

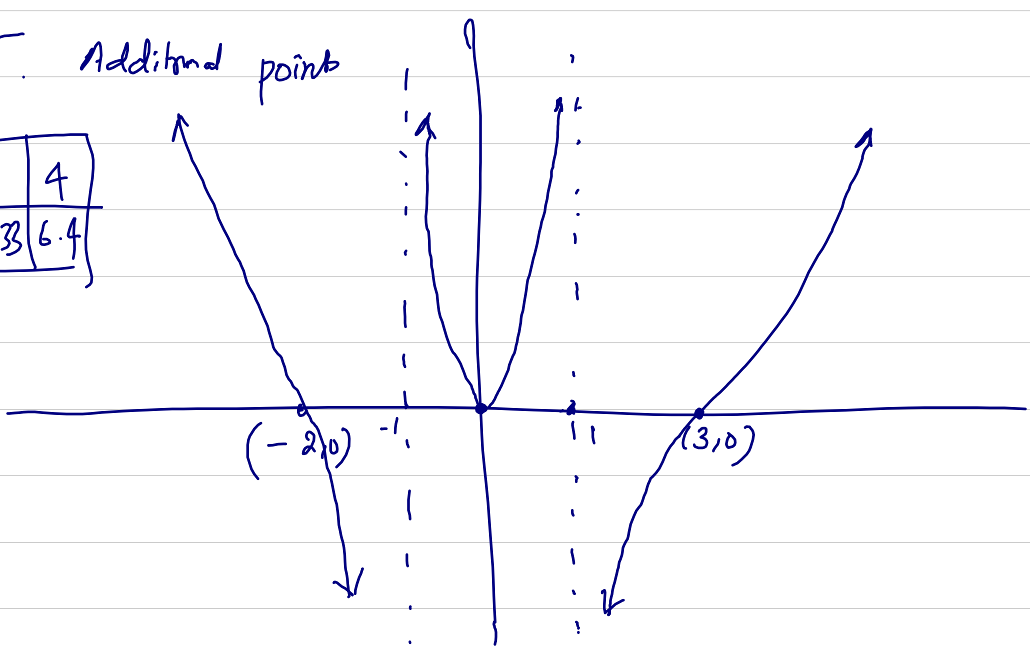
$\Rightarrow \boxed{x=1}$ and $\boxed{x=-1}$.

Horizontal: Degree of $n(x) >$ Degree of $d(x)$.
so no horizontal asymptote.

Slant: Not applicable.

Step 5. Additional points

-3	-0.5	0.5	2	4
6.75	1.75	2.08	-5.33	6.4



Exercise

State the asymptotes (if any) and graph the rational function

$$f(x) = \frac{x^3 - 2x^3 - 3x}{x+2}$$

Ex. Graph the rational function $f(x) = \frac{x^2 - 3x - 4}{x+2}$

Soln. Step 1: Domain

$x + 2 = 0$

$\Rightarrow x = -2$

$\therefore$ Domain $= (-\infty, -2) \cup (-2, \infty)$

Step 2: Intercepts

Y-int. $f(0) = \frac{0 - 0 - 4}{0 + 2} = -2.$

$\Rightarrow$ Y int $= (0, 2)$

X-int. $f(x) = 0$

$\Rightarrow \frac{x^2 - 3x - 4}{x+2} = 0$

$\Rightarrow x^2 - 3x - 4 = 0$

$\Rightarrow (x+1)(x-4) = 0$

$\Rightarrow x = -1$ and $x = 4$

$\therefore$ x-int. are $(-1, 0)$ and $(4, 0)$.

Step 3 (Holes)

$$f(x) = \frac{x^2-3x-4}{x+2}$$

$$= \frac{(x-4)(x+1)}{x+2}$$

There are no common factors, so there are no holes.

Step 4 (Asymptotes)

Vertical: $d(x) = x+2 = 0$

$$\Rightarrow x = -2.$$

$\therefore$ line $x=-2$ is Vert. asymptote.

Horizontal: $\dfrac{\text{degree of num.}}{\text{degree of denm.}} = \dfrac{2}{1}$

Thus, there is no horizontal asymptote.

Slant: degree of num. is 1 more than degree of denom.

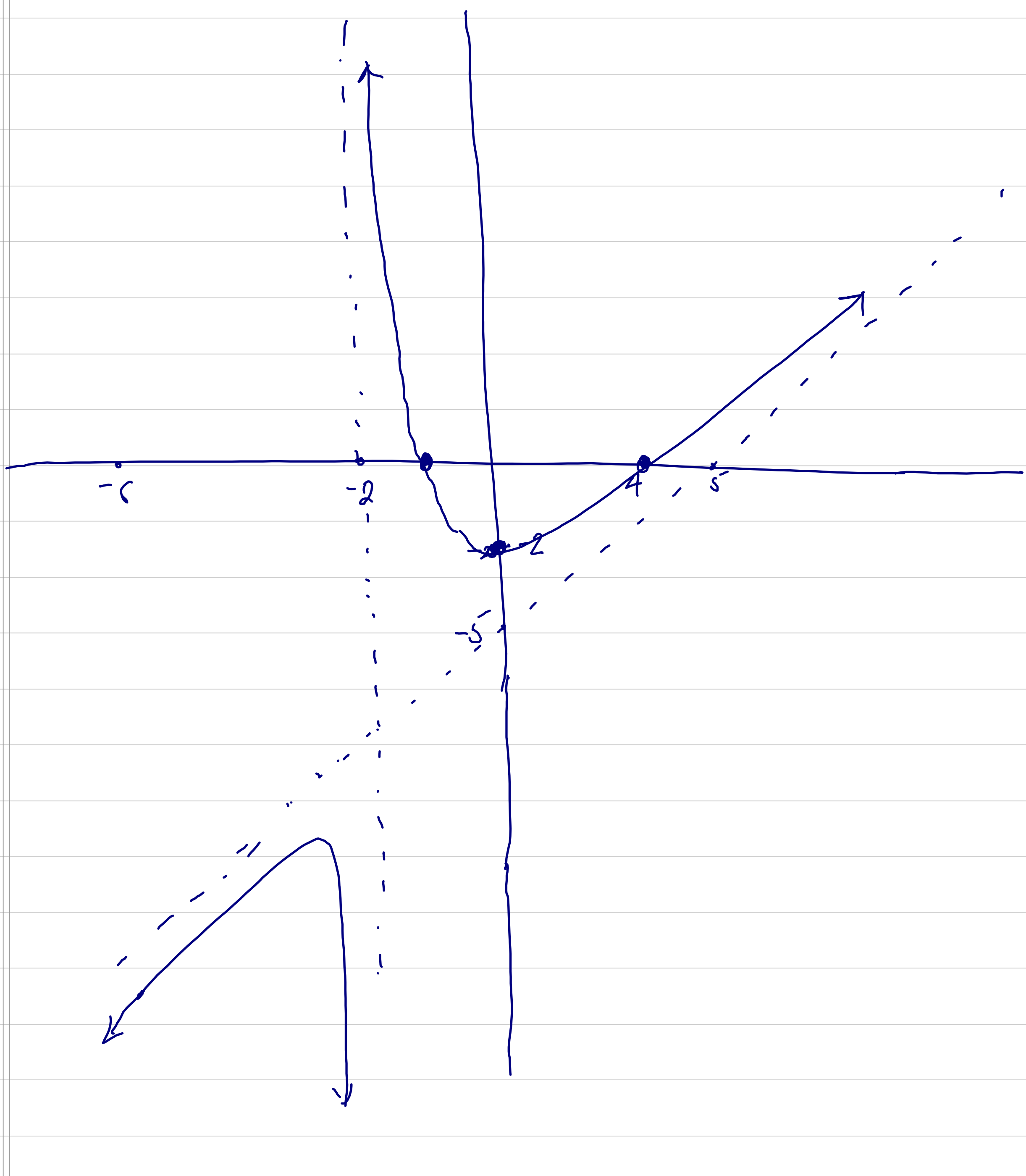
$$
\begin{array}{r|l}
x+2 \,\big)\; x^2 - 3x - 4 & x-5 \\
x^2 + 2x & \\
(-)\quad (-) & \\
\hline
-5x - 4 & \\
-5x - 10 & \\
(+)\quad (+) & \\
\hline
6 &
\end{array}
$$

$$\therefore\ f(x) = x-5+\frac{6}{x+2}$$

$\therefore$ The slant asymptote is $y = x - 5$.

Additional points:

-6	-5	-3	5	6
-12.5	-12	-14	0.86	1.75



Exercise

Graph $f(x) = \dfrac{x^2 + x - 2}{x - 3}$

Ex: Graph the rational function $f(x) = \dfrac{x^2 - x - 6}{x^2 - x - 2}$

Soln. Step 1 (Domain)

$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2 \text{ and } x = -1$$

$\therefore$ Domain $= (-\infty, -1) \cup (-1, 2) \cup (2, \infty)$

Step 2 (Intercepts)

Y-int: $f(0) = \dfrac{0^2 - 0 - 6}{0^2 - 0 - 2}$

$$= \frac{-6}{-2}$$

$$= 3$$

$\therefore$ $(0, 3)$ is Y-int.

X-int. $f(x) = 0$

$$\frac{x^2 - x - 6}{x^2 - x - 2} = 0$$

$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$
$$x = 3 \text{ and } x = -2$$

$\therefore$ $(3, 0)$ and $(-2, 0)$ are X-int.

Step 3. (Holes).

$$f(x) = \frac{\cancel{(x-2)}\,(x+3)}{\cancel{(x-2)}\,(x+1)}$$

$x-2$ is a common factor. so

$f(x) = \dfrac{x+3}{x+1}$ and $x = 2$ is a hole, i.e.

the function is undefined at $x = 2$.

Step 4 (Asymptotes)

Vertical. $\quad x+1=0$

$x = -1$

$\therefore$ the line $x = -1$ is the Vertical asymptote

Horizontal: $\dfrac{\text{Degree of num.}}{\text{Degree of denom.}} = \dfrac{2}{2} = 1$

Since the degrees of num. and denom. are equal the ratio of the leading coefficient $= \dfrac{1}{1} = 1$

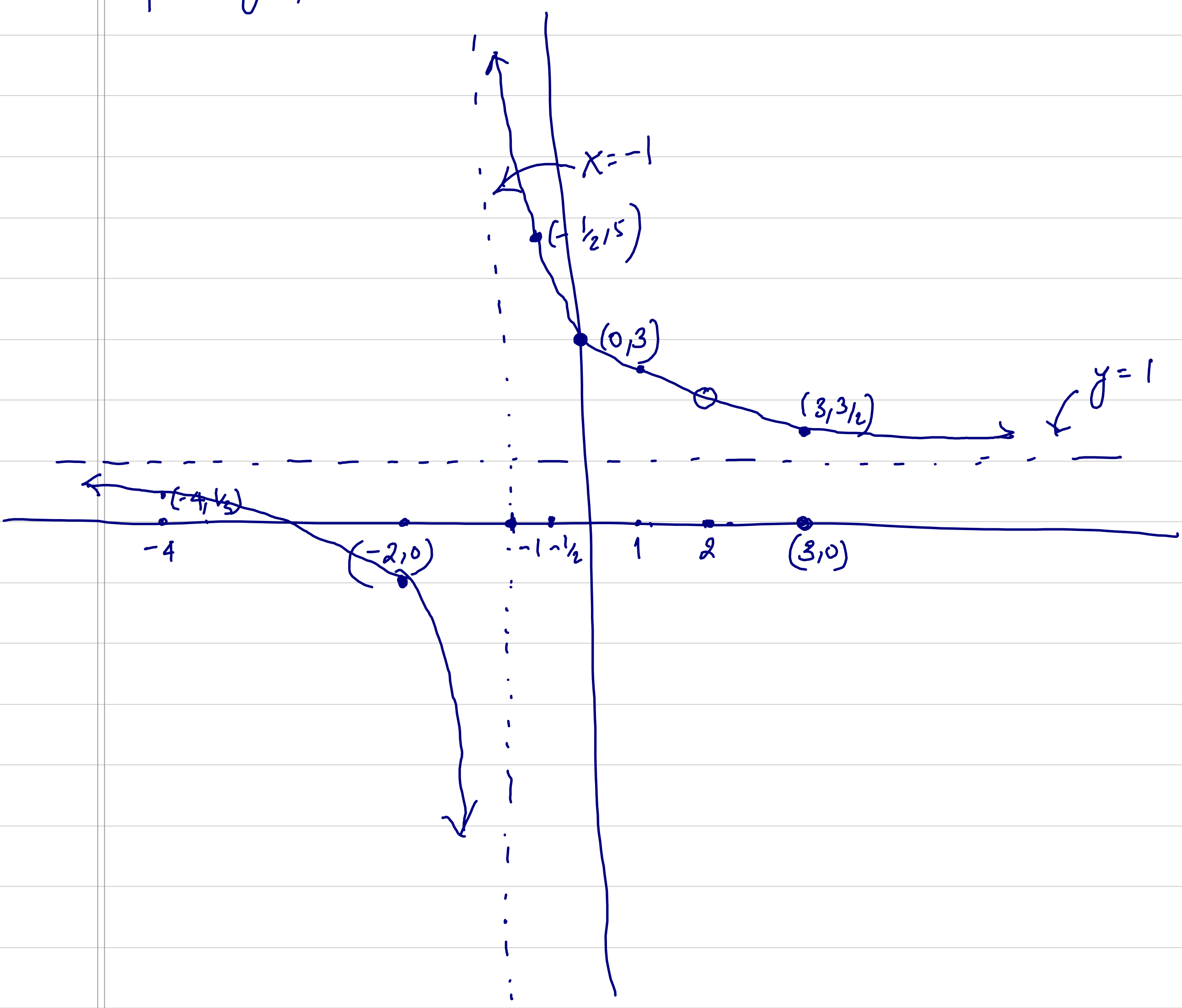
is the horizontal asymptote, i.e. the line $y = 1$ is the horizontal asymptote

Slant: Not applicable.

Step 5 Additional points

x	3	$-1/2$	-2	-4	
y	$3/2$	5	-1	$1/3$	

Step 6 graph



Exercise

Graph $f(x) = \dfrac{x^2 - x - 2}{x^2 + x - 6}$.